CALCULATING THE GASDYNAMIC AND ELECTRIC PARAMETERS OF TWO-DIMENSIONAL LOW-TEMPERATURE PLASMA FLOW IN COAXIAL MGD CHANNEL

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We propose a method for calculating the gasdynamic and electric parameters of two-dimensional conducting-gas flow in a coaxial MGD generator with continuous electrodes with account for the effect of transverse nonhomogeneity of the magnetic field, variable plasma conductivity, and dependence of the Hall parameter on the magnitude of the magnetic field and the pressure. Some computational results are presented.

A characteristic feature of the coaxial MGD generator (Fig. 1) is the essential nonhomogeneity of the magnetic field across the channel section; therefore, the problem of determining the gasdynamic and electric parameters of the flow in the channel generator must be solved in the two-dimensional formulation.

In fact, the axial flow passing through the magnetic field $B\varphi$ in the presence of the Hall effect acquires a radial displacement, since the Lorentz force in this case has two components. A similar pattern is observed in a nonhomogeneous magnetic field even without account for the Hall effect [1, 2].

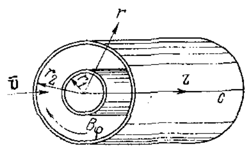

Fig. 1

In [3–5] the calculation of the coaxial channel is carried out in the quasi-one-dimensional approximation; in [5] the plasma electric parameters are averaged across the section with account for nonuniformity of the magnetic field; no estimate is given for the error of one-dimensional theory. The electric fields and currents in a coaxial Hall generator channel are studied in [6], but the problem is solved under the assumption that the plasma electric conductivity and all the gasdynamic parameters in the channel are constant. A method is described in [7] for calculating unsteady two-dimensional plasma flow in a coaxial channel and computational results are presented; however, the plasma is assumed isothermal, which is not the case in the coaxial channel of constant section.

We examine two-dimensional plasma flow in a coaxial MGD generator channel under the following assumptions:

1) the plasma is an ideal inviscid and non-heat-conducting gas,
2) the magnetic Reynolds number is small ($R_m \ll 1$), which permits neglecting the induced magnetic field,
3) the electrode-plasma contact resistance and the resistance of the electrodes can be neglected in comparison with the resistance of the plasma,
4) the entrance flow is considered uniform across the channel section,
5) end effects are not taken into account,
6) as a result of generator axial symmetry the flow parameters are independent of the angular coordinate.

Under these assumptions MGD flow of the type

$$\mathbf{V}(v_r, 0, v_z), \mathbf{B}(0, B_\varphi, 0), \mathbf{E}(E_r, 0, E_z), \mathbf{j}(j_r, 0, j_z), B_\varphi = 1/r \qquad (1)$$

is described by the following system of differential equations in dimensionless form:

$$\rho\left(v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} - j_z B_\varphi S \qquad \left(S = \frac{\sigma_0 r_1 B_0^2}{\rho_0 v_{z0}}\right)$$

$$\rho\left(v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + j_r B_\varphi S$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)+\frac{\partial}{\partial z}(\rho v_z)=0$$

$$\frac{1}{\varkappa-1}\left(v_r\frac{\partial p}{\partial r}+v_z\frac{\partial p}{\partial z}\right)-\frac{\varkappa}{\varkappa-1}\frac{p}{\rho}\left(\frac{\partial\rho}{\partial r}v_r+\frac{\partial\rho}{\partial z}v_z\right)=\frac{j_r^2+j_z^2}{\sigma},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rj_r)+\frac{\partial j_z}{\partial z}=0, \quad (2)$$

Of the Maxwell equations we use only the current continuity equation, since $R_m \ll 1$. The following characteristic dimensions are used in (2): r_1 is the inner radius of the coaxial channel; v_{z0}, ρ_0, and σ_0 are the flow parameters at the channel entrance; $B_0 = B_\varphi$ for $r = r_1$. The pressure is referred to $\rho_0 v_{z0}^2$.

The values of the current density components j_r and j_z are obtained from the generalized Ohm's Law in the form

$$j_r=\frac{\sigma}{1+\beta^2}[E_r-v_zB_\varphi+\beta(E_z+v_rB_\varphi)];$$
$$j_z=\frac{\sigma}{1+\beta^2}[E_z+v_rB_\varphi-\beta(E_r-v_zB_\varphi)], \quad (3)$$

where the electric field intensity components E_r and E_z are referred to $v_{z0}B_0$. The boundary conditions for the gasdynamic functions have the form

$$v_z=1,\ v_r=0,\ \rho=1,\ p=p_0 \text{ for } z=0,\ \ v_r=0 \text{ при } r=1,\ v_r=0 \text{ for } r=r_2. \quad (4)$$

Moreover, the condition that v_r be bounded as $z \to \infty$ must be satisfied.

The boundary conditions for the electric field intensity are formulated from the conditions that the difference of the electrode potentials be constant and the assumption of no end effects.

The problem is solved with account for the temperature and pressure dependence of the electric conductivity, which is defined by the expression

$$\sigma=AT^{3/4}e^{-\lambda/T}p^{-1/2}. \quad (5)$$

Here λ is the seed ionization potential, A is a given constant [8].

The pressure and magnetic field dependence of the Hall parameter is accounted for using the approximate formula

$$\beta=NB_\varphi/p, \quad (6)$$

where N is a constant characterizing the working medium.

The system of equations (2) with the boundary conditions (4) is solved by the small-parameter method [9]. Assuming that the magnetic interaction parameter S is small, we seek the unknown functions in the form of series expansions

$$X=X_{00}+SX_1+S^2X_2+\ldots\ (X=\mathrm{V},\ p,\ \rho,\ \sigma,\ \mathrm{E},\ \beta). \quad (7)$$

Substituting (7) into (2) and equating coefficients of like powers of S, we obtain the equations for finding the functions X_i.

As the zero approximation we use the solution of (2) for S = 0, which has the form

$$v_{z00}=1,\ v_{r00}=0,\ \rho_{00}=1,\ p_{00}=p_0,\ \sigma_{00}=1,\ \beta_{00}=\beta_0, \quad (8)$$

and as a result of account for the Hall effect in the nonuniform magnetic field the functions E_{r0} and E_{z0} are found from the equation

$$\frac{\partial}{\partial r}\frac{r}{1+\beta_0^2}(E_{r0}+\beta_0E_{z0})+\frac{r}{1+\beta_0^2}\frac{\partial}{\partial z}(E_{z0}-\beta_0E_{r0})=\frac{\partial}{\partial r}\frac{1}{1+\beta_0^2}. \quad (9)$$

Equation (9) was obtained from the current continuity equation for the zero approximation. The boundary conditions for this equation will be presented later.

Considering only terms in (7) containing the first power of S, we obtain the system of equations

$$\begin{gathered}\frac{\partial v_{r1}}{\partial z}=-\frac{\partial p_1}{\partial r}+\frac{B_\varphi}{1+\beta_0^2}[\beta_0(E_{r0}-B_\varphi)-E_{z0}],\\ \frac{\partial v_{z1}}{\partial z}=-\frac{\partial p_1}{\partial z}+\frac{B_\varphi}{1+\beta_0^2}[(E_{r0}-B_\varphi)+\beta_0E_{z0}],\\ \frac{1}{r}\frac{\partial}{\partial r}(rv_{r1})+\frac{\partial}{\partial z}(\rho_1+v_{z1})=0,\\ \frac{\partial p_1}{\partial z}-\varkappa p_0\frac{\partial \rho_1}{\partial z}=\frac{\varkappa-1}{1+\beta_0^2}[(E_{r0}-B_\varphi)^2+E_{z0}^2]\end{gathered} \tag{10}$$

and the equation

$$\frac{1}{r}q_1\frac{\partial}{\partial r}[r(E_{r1}+\beta_0E_{z1})]+q_1\frac{\partial}{\partial z}(E_{z1}-\beta_0E_{r1})+q_2(E_{r1}+\beta_0E_{z1})=q_3. \tag{11}$$

Here

$$\begin{gathered}q_1=1+\beta_0^2,\qquad q_2=-2\beta_0\frac{\partial\beta_0}{\partial r},\\ -q_3=\frac{1}{r}q_1\frac{\partial}{\partial r}\left[rB_\varphi(\beta_0v_{r1}-v_{z1})+\frac{1}{r}(2\beta_0\beta_1+\sigma_1q_1)\right]\frac{\partial}{\partial r}\{[(E_{r0}-B_\varphi)+\beta_0E_{z0}]r\}\\ +q_1\frac{\partial}{\partial z}[v_{r1}B_\varphi-\beta_1(E_{r0}-B_\varphi)+\beta_0B_\varphi v_{z1}]+q_2[B_\varphi(\beta_0v_{r1}-v_{z1})+\beta_1E_{z0}]\\ +(\beta_0E_{z0}+E_{r0}-B_\varphi)\left\{(-2)\left[\frac{\partial\beta_0}{\partial r}(\beta_1+\beta_0\sigma_1)+\beta_0\frac{\partial\beta_1}{\partial r}\right]+q_1\frac{\partial\sigma_1}{\partial r}\right\}\\ +[\beta_0(E_{r0}-B_\varphi)-E_{z0}]\left(2\beta_0\frac{\partial\beta_1}{\partial z}-q_1\frac{\partial\sigma_1}{\partial z}\right)+(\sigma_1q_1+2\beta_0\beta_1)\frac{\partial E_{z0}}{\partial z}.\end{gathered}$$

The system (10), (11) separates into two independent parts: Eqs. (10) define the functions v_{r1}, v_{z1}, p_1, and ρ_1, and include only the zero approximation E_{r0}, E_{z0}, and β_0; Eq. (11) permits finding E_{r1} and E_{z1} with account for the obtained gasdynamic functions and coefficients σ_1 and β_1. The latter are found using the following formulas, obtained by substituting (7) into (5) and (6):

$$\sigma_1=\frac{p_1}{p_0}\left(\frac{1}{4}-\frac{\lambda}{p_0^2}\right)-\frac{3}{4}\rho_1,\quad \beta_1=\frac{\beta_0p_1}{p_0},\quad \beta_0=N\frac{B_\varphi}{p_0}, \tag{12}$$

where N and λ are given constants.

Transforming (10) as indicated in [9], we obtain the equation for finding the function v_{r1},

$$(1-M_0^2)\frac{\partial^2v_{r1}}{\partial z^2}+\frac{\partial}{\partial r}\left(\frac{v_{r1}}{r}+\frac{\partial v_{r1}}{\partial r}\right)=f. \tag{13}$$

Here

$$\begin{gathered}f=(1-M_0^2)\frac{\partial f_1}{\partial z}+\frac{\partial}{\partial r}[f_3M_0^2(\varkappa-1)-f_2],\\ f_1=\frac{1}{r(1+\beta_0^2)}\left[\beta_0\left(E_{r0}-\frac{1}{r}\right)-E_{z0}\right],\qquad f_2=\frac{E_{r0}-1/r+\beta_0E_{z0}}{r(1+\beta_0^2)},\\ f_3=\frac{\varkappa-1}{1+\beta_0^2}\left[\left(E_{r0}-\frac{1}{r}\right)^2+E_{z0}\right],\qquad M_0^2=\frac{1}{\varkappa p_0}.\end{gathered}$$

The boundary conditions for (10) are

$$v_{z1}=0,\ v_{r1}=0,\ \rho_1=0,\ p_1=0\ \text{for}\ z=0,\ v_{r1}=0\ \text{for}\ r=1,\ v_{r1}=0\ \text{for}\ r=r_2. \tag{14}$$

We use the Grinberg method [10] to find the solution of (13) with the corresponding boundary conditions

$$\begin{gathered}v_{r1}=\sum u_k[J_1(\mu_kr)+C_{2k}Y_1(\mu_kr)]\left\{\int_1^{r_2}r[J_1(\mu_kr)+C_{2k}Y_1(\mu_kr)]^2dr\right\}^{-1}\\ u_k=\frac{1}{2\mu_k}\left\{\int_0^\infty F_ke^{-\mu_k(\eta+\xi)}d\eta-\int_0^\xi F_ke^{-\mu_k(\xi-\eta)}d\eta-\int_\xi^\infty F_ke^{-\mu_k(\eta-\xi)}d\eta\right\},\\ F_k=\int_1^{r_2}rf[J_1(\mu_kr)+C_{2k}Y_1(\mu_kr)]dr,\quad \xi=zC_1\quad C_1=\frac{1}{\sqrt{1-M_0^2}},\quad C_{2k}=\frac{J_1(\mu_k)}{Y_1(\mu_k)}.\end{gathered} \tag{15}$$

Here J_1 and Y_1 are Bessel functions of the first and second kind and μ_k is the k-th root of the transcendental equation

$$J_1(\mu_k)Y_1(\mu_k r) - J_1(\mu_k r)Y_1(\mu_k) = 0 \tag{16}$$

The functions v_{z1}, p_1, and ρ_1 are found from (10). Thus, considering only the first powers of S in (7), we find the gasdynamic functions from the formulas

$$\begin{gathered} v_r = Sv_{r1}, \\ v_z = 1 - \frac{S}{C_1(1-M_0^2)}\int_0^\xi \left(\frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r}\right) d\xi - \frac{S}{C_1(\varkappa p_0 - 1)}\int_0^\xi [f_3(\varkappa - 1) - f_2]\, d\xi, \\ \rho = 1 - \frac{S}{C_1}\left[\int_0^\xi \left(\frac{v_{r1}}{r} + \frac{\partial v_{r1}}{\partial r}\right) d\xi + v_{z1}\right], \\ p = p_0 + S\left(\frac{1}{C_1}\int_0^\xi f_2 d\xi - v_{z1}\right). \end{gathered} \tag{17}$$

We reduce (11) to the following form to determine the functions E_{r1} and E_{z1}:

$$\frac{\partial}{\partial r}\left[\frac{r}{q_1}(E_{r1} + \beta_0 E_{z1})\right] + \frac{rC_1}{q_1}\frac{\partial}{\partial \xi}(E_{z1} - \beta_0 E_{r1}) = \frac{rq_3}{q_1}. \tag{18}$$

Thus, (9) and (8) are obtained for finding the functions E_{r0}, E_{z0} and E_{r1}, E_{z1}; the left-hand sides of these equations have the same form, and therefore hereafter we use the indices i = 0; 1.

Introducing the electric potential φ by the formula

$$\mathbf{E} = -\operatorname{grad}\varphi,$$

and expanding φ into a series of the form (7), we obtain the expression

$$E_{ri} = -\frac{\partial \varphi_i}{\partial r}, \quad E_{zi} = -C_1\frac{\partial \varphi_i}{\partial \xi}. \tag{19}$$

Substituting (19) into (9) and (18), we reduce them to the form

$$\begin{gathered} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi_i}{\partial r}\right) + C_1^2\frac{\partial^2 \varphi_i}{\partial \xi^2} = -\frac{q_{3i}}{q_1} - \frac{C_1 q_1}{r}\frac{\partial}{\partial r}\left(\frac{r\beta_0}{q_1}\right)\frac{\partial \varphi_i}{\partial \xi} - \frac{q_3}{q_1}\frac{\partial \varphi_i}{\partial r}, \\ q_{30} = \frac{q_2}{r}, \quad q_{31} = q_3. \end{gathered} \tag{20}$$

Equations (20) can be solved by successive approximations.

As the zero approximation $\varphi_i^{(0)}$ we use the solution of (20) for $\beta_0 = 0$. In this particular case we obtain, respectively, the Laplace and Poisson equations for finding the functions $\varphi_0^{(0)}$ and $\varphi_1^{(0)}$

$$\Delta\varphi_0^{(0)} = 0, \quad \Delta\varphi_1^{(0)} = -\frac{q_{3i}}{q_1}. \tag{21}$$

We obtain the first approximation $\varphi_i^{(1)}$ by solving (20) with $\varphi_i^{(0)}$ substituted into the right-hand side, and so on.

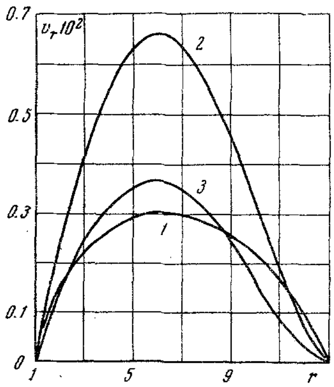

Fig. 2

We obtain the boundary conditions for the potential φ in the channel with continuous electrodes from the condition that the electrode potential difference be constant. We set

$$\varphi = 0 \text{ for } r = 1, \ \varphi = U \text{ for } r = r_2;$$

then, expanding φ into a series of the form (7), we obtain

$$\varphi_0 = 0, \ \varphi_1 = 0 \text{ for } r = 1, \ \varphi_0 = U, \ \varphi_1 = 0 \text{ for } r = r_2. \quad (22)$$

Moreover, at the channel entrance we have the condition $j_z = 0$ for $\xi = 0$, since we neglect end effects. Hence, substituting (7) into (3) we obtain

$$C_1 \frac{\partial \varphi_0}{\partial \xi} - \beta_0 \frac{\partial \varphi_0}{\partial r} = \frac{\beta_0}{r} \quad \text{for } \xi = 0, \quad (23)$$
$$\beta_0 \frac{\partial \varphi_1}{\partial r} - C_1 \frac{\partial \varphi_1}{\partial \xi} = \beta_1 \left(\frac{\partial \varphi_0}{\partial r} - \frac{1}{r} \right) \quad \text{for } \xi = 0.$$

The resulting formulas were used on an M-20 digital computer to calculate the first approximations of the gasdynamic and electric parameters

$$X = X_{00} + SX_1 \quad (X = \mathbf{V}, p, \rho, \mathbf{E}, \sigma, \beta, j). \quad (24)$$

The most characteristic relations are shown in Figs. 2–7.

All the quantities in the figures are shown in dimensionless form, and S = 0.58 and U = 0.15.

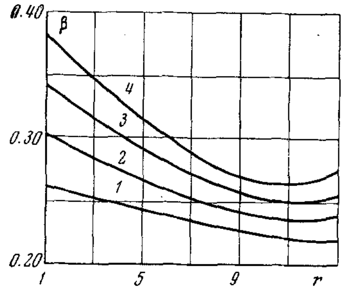

Fig. 3

Equations (9) and (11), which after transformations take the form (20), were integrated by successive approximations to determine the functions E_{z0}, E_{r0} and E_{z1}, E_{r1}. In each approximation the problem was solved by the grid method. Up to 20 successive approximations were made to achieve the specified precision (0.5%).

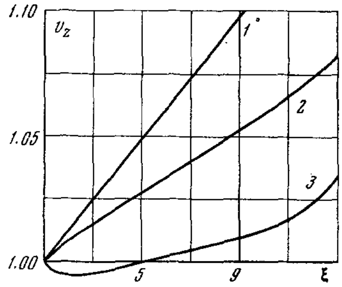

Fig. 4

We see from Fig. 2, which shows the radial velocity component v_r versus r for various values of the Hall number ($\beta = kk_1 B/p$, k = 0.5, 1), that for the coaxial channel of constant section the velocity component v_r is two orders of magnitude less than the axial velocity component v_z. However, the radial variation of the velocity v_z is rather large and comparable with its increase along the channel (Figs. 4 and 7) even without account for the Hall effect, i,e., a flow which is uniform at the channel entrance becomes essentially two-dimensional in the presence of a magnetic field which is not uniform across the channel section.

The dependence of the Hall number on the radius at different channel sections for k = 0.5 is shown in Fig. 3.

The radial velocity component arises both as a result of the existence of magnetic field nonuniformity and as a result of the presence of Hall currents. With increase of β the velocity v_r first increases and then decreases (see Fig. 2).

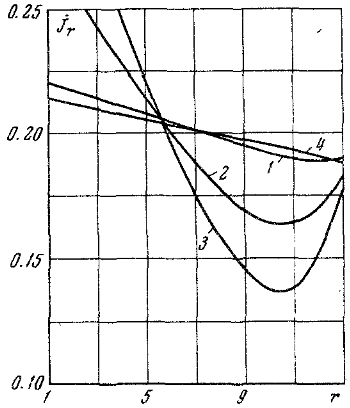

Fig. 5

Figure 4 shows curves of the dimensionless axial velocity component v_z versus the longitudinal coordinate ξ for values of k = 0, 0.5, 1. We see from comparison of curves 1, 2, and 3 that with increase of the parameter k (and this means increase of the Hall number), the relative variation of v_z along the channel length decreases. This is explained by the reduction of the flow-retarding electrodynamic force, associated basically with the reduction of the effective electric conductivity.

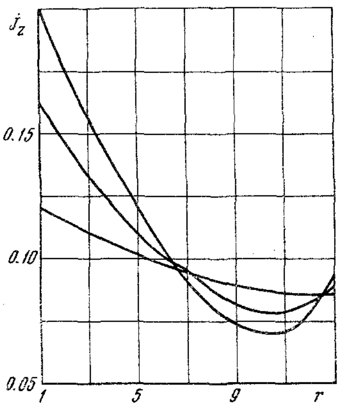

Fig. 6

The reduction of v_z in comparison with the unperturbed velocity at the channel inlet (curve 3 in Fig. 7) is explained by the strong change of the current ratio $j_r/j_z \approx \beta$. In fact, since we are considering a channel with continuous electrodes and end effects are not taken into account, the longitudinal currents j_z (we call them Hall currents) must be closed on the electrodes. Thus, the radial component j_r is the sum of the Faraday current and part of the Hall current. The Hall current which is closed by the continuous electrodes at the channel inlet closer to the outer radius exceeds the Faraday current, and this segment of the channel changes to the accelerator (pump) regime. This conclusion is also confirmed by analysis of the dependence of pressure, density, and temperature on channel length which is not presented in this article.

Analyzing the radial dependence of the current density components j_r and j_z (Figs. 5 and 6) and of the axial velocity v_z (Fig. 7), we can draw the following conclusions.

1. At the channel inlet near the inner electrode (for r = 1) the closed part of the Hall currents is directed opposite to the Faraday current, while at the end of the channel their directions coincide, therefore in segment 1-5 (along the Or-axis) curve 3 corresponding to the section $\xi = \xi_{10}$ lies above curves 1 and 2 ($\xi = \xi_2$ and $\xi = \xi_6$, respectively). Near the inner electrode we observe the reverse direction of the closed part of the Hall currents, therefore in segment 5-13 curve 3 lies below curves 1 and 2 (Fig. 5).

2. Since the longitudinal current arises basically as a result of interaction of the resultant radial current with the magnetic field, the nature of the radial variation of j_z should coincide with the nature of the variation of j_r, and this is shown by comparison of Figs. 5 and 6.

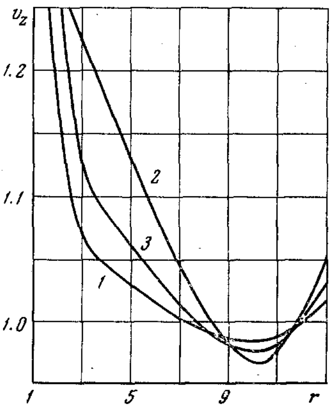

Fig. 7

3. We see from Fig. 7 that when the corresponding Hall number is reached near the inner electrode (in segment 1-2) there arises a regime of very strong increase of the longitudinal velocity component, which apparently corresponds to the "oscillation" regime at the anode [7].

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